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STRESS DISTRIBUTION DETERMINATION IN ISOTROPIC MATERIALS VIA INVERSION OF ULTRASONIC RAYLEIGH WAVE DISPERSION DATA

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Abstract—The possibility of using measurements of the dispersion of Rayleigh (surface) waves propagating on pre-stressed, initially isotropic materials to determine the surface stress and gradients in such materials is analytically investigated. Using a first-order perturbation formula for the description of the acoustoelastic effect on Rayleigh waves, it is shown that knowledge of the frequency dependence of the change in phase (or phase velocity) of the Rayleigh wave after propagating a certain distance can, in theory, be used to determine the stress level at the surface of the medium as well as its derivatives up to a given order. Basic questions concerning the uniqueness of the inversion are addressed, and a formal method of inverting inhomogeneous bi-axial stress distributions is presented. The effect of the range of frequencies included in the measurements is discussed and illustrated with a numerical example. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

The effect of an initial, possibly finite, deformation of a body on the equations governing the propagation of small amplitude elastic waves in the body has been extensively analyzed, dating back to the time of Cauchy (1828). The so-called acoustoelastic effect has since emerged from such investigations. The acoustoelastic effect refers to the fact that an initial deformation of a body causes a (small) change in the velocity of propagation of elastic waves through the body. Accurate measurements of the stress-induced velocity changes yield information which can, in some cases, lead to a determination of the stresses in the body. Alternatively, one can make use of the fact that initial deformation (or stress) causes an apparent anisotropy in an initially isotropic medium, destroying the degeneracy of shear wave speeds. In stressed materials, therefore, two shear waves with different polarization directions will, in general, propagate at different speeds. The difference in speed of two orthogonally polarized shear waves is proportional to the difference in the principal stresses, and the polarization directions of the two waves coincide, in initially isotropic media, to the directions of the principal stresses. Because of its strong similarity to photoelasticity, such a phenomenon has been called acoustoelasticity. The difference in speed of orthogonally polarized shear waves causes the two waves, which may have been initially in phase, to gradually develop a phase difference. The measurement of such a phase difference is much easier than a direct measurement of velocity changes which are, in almost all cases, extremely small. Hirao *et al.* (1992) have used resonance measurements instead of direct or differential velocity measurements to increase accuracies for examination of specimens of small thickness.

The acoustoelastic effect can be observed in the propagation of virtually all types of elastic waves including bulk longitudinal and shear waves as well as guided surface (Rayleigh) and plate (Lamb) waves. Most approaches to predicting the effect of a pre-stress (or deformation) on the propagation of small amplitude elastic waves start with a modified form of Navier's displacement equations of motion which include in them terms corresponding to the initial deformation field, i.e., the initial strains. For homogeneous initial deformations the resulting governing equations have constant coefficients and can be solved exactly for various types of assumed solutions corresponding to, for example, plane longitudinal and shear waves, (Murnaghan (1951), Hughes and Kelley (1951), Toupin and Bernstein (1961), Hayes and Rivlin (1961a), Tokuoka and Iwashimizu (1968), and Husson and Kino (1982)) as well as Love (Hayes and Rivlin (1961b)) and Rayleigh waves (Hirao, *et al.* (1981), Tverdokhlebov (1983) and Husson (1985)). The effect of pre-stress on the existence and propagation characteristics of interface waves in a layer embedded in an infinite space was also recently examined (Sotiropoulos and Sifniotopoulos (1995)) as was the case of Rayleigh wave propagation in a pre-stressed layer attached to an underlying pre-stressed half-space (Ogden and Sotiropoulos (1995)). For non-uniform but spatially continuous initial deformation however, the governing partial differential equations have spatially non-constant coefficients and become exceedingly difficult to solve. For this reason, acoustoelasticity theory is much less developed for such cases.

One of the few attempts at predicting the effects of a non-homogeneous initial deformation on the propagation of surface waves is that of Hirao *et al.* (1981). They examined cases of uniform and linear uniaxial initial stress distributions which varied only with depth into the half space on which the surface wave was assumed to propagate. They also performed experiments on bent plates to verify the theory. A major result of their work was the theoretical and experimental verification of the fact that Rayleigh waves, which are non dispersive on homogeneous, non-stressed or uniformly stressed media, become dispersive when propagating on a medium which contains a non-uniform stress distribution.

Husson and Kino (1982) took a different approach to predicting the effect of a nonhomogeneous initial deformation on the propagation of bulk waves. In their approach, based on "energy perturbation methods", the medium with initial stress was considered a perturbation of the corresponding stress free medium and an integral relation was obtained relating the fields in the perturbed and unperturbed media. A first order Born approximation (i.e., the particle velocity field distribution in the stressed medium is assumed to be the same as that in the unstressed medium) was then made to allow the integral relation to be simplified into an expression for the change of phase of the wave due to the presence of the applied stress field. Husson (1985) later applied this work to the case of Rayleigh and Lamb waves propagating on and in inhomogeneously stressed media. A major advantage of this latter approach is that determination of the effect of the initial, possibly inhomogeneous, stress field on the propagation of surface waves requires only information about the gradients of the displacement field of the Rayleigh wave in the unstressed medium. This latter work by Husson is the starting point of the present work.

In the next section, some brief background material will be given on the Husson-Kino approach to acoustoelasticity of Rayleigh waves. In the sections following, the formal method of inverting the dispersion information to obtain the stress distribution will be detailed.

BACKGROUND

Husson (1985), based on work by Husson and Kino (1982), derived a formula for the change of phase of a small amplitude Rayleigh wave due to propagation over the surface of an initially stressed isotropic solid. He showed that a Rayleigh wave with initial particle velocity and stress fields $\mathbf{v}$ and $\mathbf{T}$ respectively, would attain a phase shift of $\phi(\omega)$ and hence become $\mathbf{v}e^{i\phi}$ and $\mathbf{T}e^{i\phi}$ after propagating a given distance over the surface of the material. Denoting by ϕ^0 the phase shift which would have been experienced by the wave had it propagated on an equivalent stress-free medium, and by $\delta\phi$ the difference $\phi-\phi^0$, Husson arrived at an expression for $\delta\phi$ in the form:

$$\delta\phi = -\frac{\omega}{4P}\int_V G\,\mathrm{d}V. \tag{1}$$

In eqn (1), ω denotes circular frequency and V denotes a volume enclosing the Rayleigh wave, which may be a confined beam or an infinitely extended plane wave. We deal here only with the case of a plane crested Rayleigh wave with fronts which are infinitely extended in the direction perpendicular to the propagation direction. P denotes the average over one time period of the power carried by the Rayleigh wave per unit width in the direction perpendicular to the travel direction. As shown by Auld (1990), it can be expressed in terms of the product of the group velocity of the Rayleigh wave and its kinetic energy density. Using notation similar to that of Husson (1985), one can write

$$P = \frac{\omega\rho_o V_o}{2}\left[\frac{1/V_o^2+\kappa_s^2}{2\kappa_s} - 2\frac{K_2/V_o^2+K_4\kappa_s^2}{\kappa_s+\kappa_l} + \frac{K_2^2/V_o^2+K_4^2\kappa_s^2}{2\kappa_l}\right], \tag{2}$$

where the following quantities have been defined:

$$\kappa_s \equiv \left(\frac{1}{V_o^2} - \frac{1}{V_s^2}\right)^{1/2}, \tag{3a}$$

$$\kappa_l \equiv \left(\frac{1}{V_o^2} - \frac{1}{V_l^2}\right)^{1/2}, \tag{3b}$$

$$K_2 \equiv \frac{2\kappa_s\kappa_l}{[1/V_o^2+\kappa_s^2]}, \tag{3c}$$

$$K_4 \equiv \frac{2}{[1+V_o^2\kappa_s^2]}. \tag{3d}$$

In eqns (3a–d), V_o, V_s and V_l denote the phase velocities of the Rayleigh, transverse and longitudinal waves respectively in the unstressed medium. ρ_o denotes the constant density of the medium before application of the static initial deformation.

Finally, the integrand of eqn (1) can be expressed in terms of the initial deformation gradients in the medium, the second (λ, μ) and third (l, m, n) order elastic constants of the medium and the displacement gradients caused by the Rayleigh wave. Denoting by b_i, $i \in \{1, 2, 3\}$ the components of the initial static displacements of the medium and by a_i, $i \in \{1, 2, 3\}$ the coordinates of a material particle in the undeformed state, G can be expressed by:

$$\begin{aligned} G \equiv {} & \frac{\partial b_m}{\partial a_m}\{(2l+\lambda)[A(a_2,\omega)+B(a_2,\omega)+C(a_2,\omega)]+(\lambda+m)D(a_2,\omega)+mE(a_2,\omega)\} \\ & + \frac{\partial b_2}{\partial a_2}\{(2\lambda+6\mu+4m)A(a_2,\omega)+\mu[2D(a_2,\omega)+E(a_2,\omega)]\} \\ & + \frac{\partial b_3}{\partial a_3}\{(2\lambda+6\mu+4m)B(a_2,\omega)+\mu[2D(a_2,\omega)+E(a_2,\omega)]\} \\ & - \frac{\partial b_1}{\partial a_1}\{(n/2)[D(a_2,\omega)+E(a_2,\omega)]+(\lambda+2m-n)C(a_2,\omega)\}, \end{aligned} \tag{4}$$

where the functions $A(a_2, \omega)$ through $E(a_2, \omega)$ depend upon the gradients of the displacement field of the Rayleigh wave when propagating on the equivalent unstressed medium. We note that eqn (4) differs slightly from the expression originally given by Husson (1985). We have verified through personal contact with Husson that an algebraic error exists in the original expression. There are several typos in the original expressions for the functions A through E, hence these functions are given in slightly rearranged and corrected form later (see eqn (7) and the Appendix). The second order elastic constants of the medium are taken as the Lamé constants λ and μ, and the third order constants used are those of Murnaghan, l, m and n. Summation over repeated subscripts is implied in (4) and throughout.

In deriving eqn (4), the Rayleigh wave was assumed to propagate in the a_3 direction which, together with the a_2 direction defines the sagittal plane which contains the polarization vector of the Rayleigh wave. The coordinate a_1 is perpendicular to the saggital plane and consequently the a_1 component of the Rayleigh wave displacement vanishes and all fields are independent of a_1. In addition, only the diagonal terms of the displacement gradient tensor of the initial deformation have been taken into account. This implies that eqn (4) is applicable only to cases where the initial shear deformation components vanish throughout the region encountered by the Rayleigh wave. The presence of specific non-diagonal terms of the initial displacement gradient tensor would cause the Rayleigh wave to undergo additional phase changes during propagation. The original paper by Husson and Kino (1982) examines in detail the effect of such terms. It should be noted that since the Rayleigh wave propagates on a traction free surface, for large enough frequencies where the wave is confined to a small region about the surface, eqn (4) will be approximately true even if there is initial shear deformation in the interior of the solid. This follows from the fact that those components of shear which affect the Rayleigh wave would necessarily vanish at and, due to continuity, near the free surface.

Equations (1–4) allow the solution of the so-called direct problem; i.e., that of predicting the change of phase of a Rayleigh wave caused by an initial deformation (or stress). Before presenting an example of the application of the formula, it is specialized to some specific cases which will be of importance in later developments. First of all, the Rayleigh wave is assumed to propagate in the a_3 direction over a length L_0, and to have uniform fields in the a_1 direction. If a uniaxial normal stress which varies only with depth is applied along the a_3 axis, $(\sigma_{33}(a_2))$, and Hooke's law is used to relate the initial stresses and strains, the initial displacement gradients assume the forms,

$$\frac{\partial b_3}{\partial a_3} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_{33} \tag{5}$$
$$\frac{\partial b_1}{\partial a_1} = \frac{\partial b_2}{\partial a_2} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{33}.$$

Under this assumption, eqn (1) reduces to the form:

$$\delta\phi^{33}(\omega) = -\frac{L_0\omega}{4P}\int_0^\infty \alpha_i^{\|} F_i(a_2,\omega)\sigma_{33}(a_2)\,\mathrm{d}a_2, \tag{6}$$

where summation over the repeated index $i \in \{1,\ldots,5\}$ is implied here and throughout. To make the notation more compact, the original functions $A(a_2,\omega)$ through $E(a_2,\omega)$ appearing in (4) have been relabelled $F_i(a_2,\omega) i \in \{1,\ldots,5\}$. This allows use of the summation convention to imply summation over repeated subscripts. Each of these functions can be written in the general form:

$$F_i(a_2,\omega) = \omega^2\{f_{i1}e^{-2\omega\kappa_s a_2} + f_{i2}e^{-2\omega\kappa_l a_2} + f_{i3}e^{-\omega(\kappa_l+\kappa_s)a_2}\}, \tag{7}$$

where the 15 constants f_{ij}, $i \in \{1,\ldots,5\}, j \in \{1,2,3\}$ are given in the Appendix.

The constants $\alpha_i^{\|}$ appearing in eqn (6) are given by

$$\alpha_1^{\parallel} \equiv \frac{1}{(3\lambda+2\mu)}\left\{\lambda+2l-\frac{\lambda(2\lambda+6\mu+4m)}{2\mu}\right\},$$

$$\alpha_2^{\parallel} \equiv \frac{1}{(3\lambda+2\mu)}\left\{\lambda+2l+\frac{(\lambda+\mu)(2\lambda+6\mu+4m)}{\mu}\right\},$$

$$\alpha_3^{\parallel} \equiv \frac{1}{(3\lambda+2\mu)}\left\{\lambda+2l+\frac{\lambda(\lambda+2m-n)}{2\mu}\right\},$$

$$\alpha_4^{\parallel} \equiv \frac{1}{(3\lambda+2\mu)}\left\{3\lambda+2\mu+m-\frac{\lambda(2\mu-n/2)}{2\mu}\right\},$$

$$\alpha_5^{\parallel} \equiv \frac{1}{(3\lambda+2\mu)}\left\{\lambda+\mu+m-\frac{\lambda(\mu-n/2)}{2\mu}\right\}. \tag{8}$$

The notation $\delta\phi^{\alpha\beta}(\omega)$ is used in eqn (6) and throughout to mean the possibly frequency dependent change in phase experienced by a Rayleigh wave propagating in the a_α direction caused by a uniaxial stress applied in the a_β direction. The notation $\delta\phi^{\alpha(\beta+\gamma)}$ is used to denote the change in phase experienced by a Rayleigh wave propagating in the a_α direction caused by a bi-axial stress applied with (orthogonal) principal directions along the a_β and a_γ directions.

For an applied uniaxial stress in the a_1 direction and propagation in the a_3 direction, the initial displacement gradients can be expressed as

$$\frac{\partial b_1}{\partial a_1} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_{11} \tag{9}$$

$$\frac{\partial b_2}{\partial a_2} = \frac{\partial b_3}{\partial a_3} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{11}.$$

Equation (1) then specializes to

$$\delta\phi^{31}(\omega) = -\frac{L_0\omega}{4P}\int_0^{\infty}\alpha_i^{\perp}F_i(a_2,\omega)\sigma_{11}(a_2)\,\mathrm{d}a_2, \tag{10}$$

where five additional constants, $\alpha_i^{\perp}$, depending only on the second and third order elastic constants of the medium, are given by

$$\alpha_1^{\perp} \equiv \alpha_1^{\parallel},$$

$$\alpha_2^{\perp} \equiv \alpha_1^{\perp},$$

$$\alpha_3^{\perp} \equiv \frac{1}{(3\lambda+2\mu)}\left\{\lambda+2l-\frac{(\lambda+\mu)(\lambda+2m-n)}{\mu}\right\},$$

$$\alpha_4^{\perp} \equiv \frac{1}{(3\lambda+2\mu)}\left\{m-\lambda-\frac{(\lambda+\mu)n}{2\mu}\right\},$$

$$\alpha_5^{\perp} \equiv \alpha_4^{\perp}. \tag{11}$$

Two very important aspects of eqn (1) and its specialization in eqns (6) and (10) are: (1) in this first order treatment of the acoustoelastic effect, the change of phase is a linear functional of the applied stress distribution; and (2) the phase of the wave is affected by the stress both parallel and perpendicular to the direction of propagation. These two facts will be shown to introduce an inherent difficulty when trying to determine a bi-axial stress field using dispersion data for propagation in a single direction.

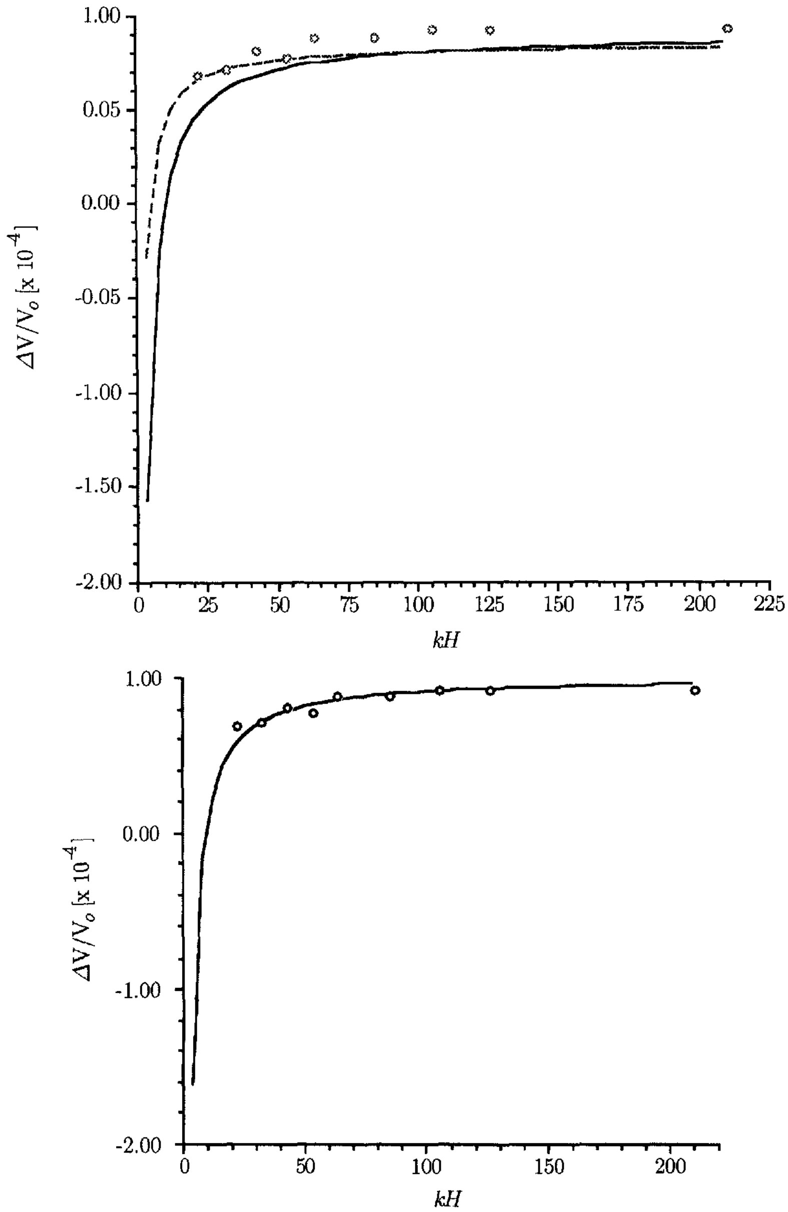

Fig. 1. (a) Plots of the relative change in phase velocity of a surface wave propagating perpendicularly to an initial applied stress distribution which varies linearly with depth. Dashed line, from Hirao *et al.* (1981); solid line, present theory eqn (10); open circles, experiments from Hirao *et al.* (1981). (b) Same as (a), but with slightly adjusted third order elastic constants. Solid line, present theory; open circles, experiments from Hirao *et al.* (1981).

As an example of the application of the above formulae, we ran a case similar to that run by Hirao *et al.* (1981). The results, corresponding to the propagation of a Rayleigh wave on a mild steel bar with elastic constants, $\lambda = 107.4$, $\mu = 81.9$, $l = -206.5$, $m = -600$ and $n = -800$ [GPa], containing a uniaxial, linear stress distribution perpendicular to the propagation direction are shown in Fig. (1a). The initial static stress was assumed in the form, $\sigma_{11}(a_2) = \bar{\sigma}_{11}(1-2a_2/H)$ with $\bar{\sigma}_{11} = 0.06375$ [GPa] and $H = 10$ mm. The initial stress thus equals $\bar{\sigma}_{11}$ when $a_2 = 0$, it vanishes when $a_2 = H/2$ and it equals $-\bar{\sigma}_{11}$ when $a_2 = H$. This is the same stress distribution which would be obtained if a plate of thickness H were bent by couples applied to the ends, which Hirao *et al.* (1981) used to compare theory to experiment.

Shown in Fig. 1a is a plot of the change in phase velocity vs wavenumber ($k \equiv \omega/V_o$, i.e., the wavenumber of the Rayleigh wave on an unstressed medium) times plate thickness,

H. The results for the change in phase can, for propagation in the a_3 direction, be converted to change in phase velocity using the formula:

$$\frac{\Delta V}{V_0} = \bar{\varepsilon}_{33} - \frac{\delta\phi^{33} V_0}{\omega L_0} \tag{12}$$

where $\bar{\varepsilon}_{33}$ denotes the surface strain in the direction of propagation. In both the current approach and that of Hirao *et al.*, the resulting expression for the change of phase velocity can be written in the form

$$\frac{\Delta V}{V_0} = \beta_0 \bar{\sigma}_{11} + \frac{\beta_1}{\omega} \overline{\left(\frac{\partial \sigma_{11}}{\partial a_2}\right)} \tag{13}$$

for suitable values of the constants β_0 and β_1. The overbar denotes values evaluated at the free surface, $a_2 = 0$. As can be seen by eqn (13), the phase velocity of the surface wave is affected by both the surface stress and its gradient at the surface. The non-uniform initial stress distribution is seen to cause dispersion of the Rayleigh wave via the second term in (13).

Figure 1a shows that there is a slight discrepancy between the two approaches for low frequency, the cause of which we are not exactly sure at this point. The values predicted by the two theories become closer with increasing frequency. Also shown in Fig. (1a) as open circles are the experimental data reported by Hirao *et al.* (1981). As mentioned by Hirao *et al.*, the accuracy of determining the third order elastic constants is not great. We therefore reran our analysis with slightly modified values of the third order elastic constants. The result for $l = -191.5$, $m = -585.0$ and $n = -800$ is shown in Fig. 1b along with the experimental data of Hirao *et al.* Using this modified set of elastic constants, which corresponds to a 7.5% in one of the constants determined by Hirao *et al.* (their v_2), the theoretical data from the current theory is seen to match the experimental data remarkably well.

INVERSE PROBLEM

Uniaxial stress states

The formula, eqn (1), and its specializations in eqns (6) and (10) relate the change in phase of the Rayleigh wave to the applied or residual stress in the medium. Considering the change in phase (as a function of frequency) as measurable by suitable experimental means, the left hand sides of the equations represent known functions of frequency. Equations (6) and (10) can then be classified as Fredholm integral equations of the first kind for the unknown stress distributions $\sigma_{33}(a_2)$ or $\sigma_{11}(a_2)$ respectively (Hochstadt (1989)). While the theory of solving such equations is relatively well developed for certain types of equations, such as equations of the second kind, or for specific types of kernels, in general, not much can be said concerning the general kernels which appear in these equations (see eqn (7)). Furthermore, any solutions obtained from rigorous mathematical methods are unlikely to be of much use in the practical solution of these equations since the left hand side of these equations will not in general be known for all frequencies. Rather, the change of phase of the Rayleigh wave will invariably be measured over a finite frequency range. A very important practical question therefore is whether eqns (6) and (10) can be used to estimate the stress distributions for limited data on the (presumably known) functions $\delta\phi^{\alpha\beta}(\omega)$.

In this section an attempt is made at inversion of these equations by a method similar to Enskog's method of solving Fredholm equations of the second kind (Tricomi (1985)). The essence of the method is to assume that the unknown function under the integral can be expanded in a series of functions of suitable form with a-priori unknown coefficients. Substitution of the assumed series under the integral and subsequent term-by-term integration (if possible) converts the integral into another series of functions. If the known inhomogeneous term (i.e., the left hand sides in eqns (6) and (10)) can be expanded uniquely in a series of these resulting functions, one has only to match coefficients of the expansions to determine the coefficients of the original series expansion of the unknown function.

Proceeding, assume that a uniaxial normal stress $\sigma_{33}(a_2)$ exists in a medium and that this stress distribution can be sufficiently approximated by a truncated Maclaurin series (i.e., an N^{th} order Maclaurin polynomial) about the free surface, $a_2 = 0$ in the form,

$$\sigma_{33}(a_2) = \sum_{n=0}^{N} \sigma_n^{33} a_2^n \tag{14}$$

with the general coefficient, σ_n^{33}, being proportional to the derivative of σ_{33} at the surface, $(\mathrm{d}^n\sigma_3(0)/\mathrm{d}a_2^n)/n!$ While certainly not all functions will admit an accurate representation of the form given in (14), Weierstrass' theorem (Achieser (1992)) proves that any continuous function can be approximated on a finite interval to any degree of accuracy by a sufficiently high order polynomial of the form given in (14). If the initial stress is continuous and vanishes beyond a certain depth, Weierstrass' theorem can be invoked to show that an expansion of the form given in (14) is, for some possibly large value of N, an accurate approximation.

Assuming propagation along the applied stress direction (a_3) the expansion, eqn (14), is substituted into eqn (6) to obtain the change in phase of the wave. When this is done there results

$$-\frac{4(P/\omega)\delta\phi^{33}(\omega)}{L_0} = \alpha_i^l \sum_{n=0}^{N} \sigma_n^{33} \int_0^{\infty} F_i(a_2,\omega) a_2^n \,\mathrm{d}a_2 \tag{15}$$

where again, summation over $i \in \{1, \ldots, 5\}$ is implied. An interchange of integration and summation was performed to arrive at eqn (15), which is valid since there is a finite number of terms in the summation. Note that since P is proportional to ω (see eqn (2)), the term (P/ω) on the left is independent of frequency.

Using the expression for the $F_i(a_2, \omega)$ given in eqn (7), and the well known result (Gradshteyn and Ryzhik (1994)),

$$\int_0^{\infty} a_2^n e^{-\omega a_2} \,\mathrm{d}a_2 = \frac{\Gamma(n+1)}{\omega^{n+1}}, \tag{16}$$

where $\Gamma(n+1) = n!$ represents the Gamma function, the integrals appearing in eqn (15) can be written for arbitrary n in the form

$$\int_0^{\infty} F_i(a_2,\omega) a_2^n \,\mathrm{d}a_2 = \frac{\Gamma(n+1)}{\omega^{n-1}} \left\{ \frac{f_{i1}}{(2\kappa_s)^{n+1}} + \frac{f_{i2}}{(2\kappa_l)^{n+1}} + \frac{f_{i3}}{(\kappa_s+\kappa_l)^{n+1}} \right\} \equiv \frac{C_{in}}{\omega^{n-1}} \tag{17}$$

where the C_{in}, $i \in \{1, \ldots, 5\}$; $n \in \{0, \ldots, N\}$ have been defined as:

$$C_{in} \equiv n! \left\{ \frac{f_{i1}}{(2\kappa_s)^{n+1}} + \frac{f_{i2}}{(2\kappa_l)^{n+1}} + \frac{f_{i3}}{(\kappa_s+\kappa_l)^{n+1}} \right\}, \tag{18}$$

with the fifteen constants, f_{ij}, given in the Appendix. With these definitions, eqn (15) can be written in the compact form:

$$-\frac{4(P/\omega)\delta\phi^{33}(\omega)}{L_0} = \alpha_i^{\|} \sum_{n=0}^{N} \frac{C_{in}\sigma_n^{33}}{\omega^{n-1}}. \tag{19}$$

It can be seen from eqn (19) that substitution of the Maclaurin series expansion for the unknown stress distribution leads to a series of functions of powers of frequency ω. Recall that the quantity (P/ω) is actually independent of frequency. Of prime importance is the fact that the resulting series of functions of ω are linearly independent. The Wronskian of any two of these functions, say $1/\omega^{n_1-1}$ and $1/\omega^{n_2-1}$, for arbitrary integers n_1 and n_2 equals $\pm(n_1-n_2)\omega^{1-(n_1+n_2)}$ and hence vanishes only if $n_1 = n_2$. This allows us to expand suitable functions uniquely in terms of this set of functions. As discussed by Tricomi (1985), the only time Enskog's method fails is when the original (lineary independent) series of functions are mapped by the integral operator into a linearly dependent set. This is not the case for the integral operators defined in eqns (6) and (10).

It should at this point be noted, however, that not all functions of ω are expandable in terms of the functions given in eqn (19). That is, those functions are not complete in, say, the space of square integrable or even continuous functions on $\omega \in [0,\infty)$. Mathematically, one of the reasons that the resulting functions of ω are not complete (again, in some space) is the fact that the original series expansion of the stress was also in terms of a non-complete (finite) set of functions. Physically, the fact that the measured $\delta\phi(\omega)$ may not be expandable in terms of the functions in eqn (19) would signify that the stress distribution in the medium is not expandable in the series shown in eqn (14). As mentioned earlier, however, any continuous function can be approximated, on any finite interval and to any degree of approximation by a polynomial of the form given in (14). If the initial stress is continuous, therefore, the measured $\delta\phi(\omega)$ will be expressible in the form of eqn (19).

The change of phase, $\delta\phi^{33}(\omega)$ (or equivalently, the change in phase velocity) of the wave can, in principle, be measured experimentally as a function of frequency (Allen and Cooper (1983)). Assuming that this is done, one can find the coefficients, Φ_n^{33}, of an expansion of the form:

$$-\frac{4(P/\omega)\delta\phi^{33}(\omega)}{L_0} = \Phi_{-1}^{33}\omega + \Phi_0^{33} + \frac{\Phi_1^{33}}{\omega} + \ldots + \frac{\Phi_{N-1}^{33}}{\omega^{N-1}} \tag{20a}$$

or, more compactly,

$$-\frac{4(P/\omega)\delta\phi^{33}(\omega)}{L_0} = \sum_{n=-1}^{N-1} \frac{\Phi_n^{33}}{\omega^n} \tag{20b}$$

which will give a best fit to the experimentally obtained data. One could, of course, solve explicitly for the coefficients Φ_n^{33} by equating the expansion, eqn (20a), to the experimentally obtained data for $N+1$ frequencies, ω_k, $k \in \{1, 2, \ldots, N+1\}$. The effect of slight measurement errors will, however, be greatly reduced if some type of least squares, non-linear curve fitting algorithm is used to determine the expansion coefficients, Φ_n^{33} from the measured $\delta\phi^{33}(\omega)$ data using a redundant number of frequencies.

In any case, assuming that the coefficients Φ_n^{33} for $n \in \{-1, \ldots, N-1\}$ have been obtained, eqns (19) and (20) give a relation between the then known coefficients, Φ_n^{33} and the sought after stress expansion coefficients, $\sigma_n^{33} n \in \{0, \ldots, N\}$. Comparing terms of like powers of ω in the expansions (19) and (20b), one has

$$\sigma_n^{33} = \frac{\Phi_{n-1}^{33}}{\alpha_i^{|} C_{in}} \quad \text{(no sum on } n\text{)}, \tag{21}$$

where the summation convention applies to the denominator. Equation (21) gives the formal relation between the experimentally obtained coefficients, Φ_{n-1}^{33}, and the gradients of the applied/residual stress field, σ_n^{33}. It was obtained under the assumption of propagation along the direction of the uniaxial stress $\sigma_{33}(a_2)$. Equation (21) can in principle be used to determine the surface stress, σ_0^{33} as well as any of the gradients of the stress at the surface so long as the appropriate coefficient, $Q_n^1 \equiv \alpha_i^{\parallel} C_{in} \neq 0$.

It is important to note that the range of frequencies included in the experimental data are of critical importance to the success of the stress determination. This can be seen by direct examination of eqn (19). In fact, using the relation between $\Delta V/V_0$ and $\delta\phi^{33}$, eqn (12), in addition to eqn (19) and the relation $\bar{\varepsilon}_{33} = (1/E)\bar{\sigma}_{33} = (1/E)\sigma_o^{33}$, it can be seen that:

$$\lim_{\omega\to\infty} \frac{\Delta V}{V_0} = \left\{\frac{1}{E} + \frac{V_0 \alpha_i^{\parallel} C_{i0}}{4(P/\omega)}\right\}\sigma_o^{33}, \tag{22}$$

i.e., only the value of the surface stress affects the phase velocity of the Rayleigh wave at high enough frequencies. Examining eqn (20a), it can be seen that the choice of frequency range greatly affects which coefficients, Φ_n^{33}, contribute to the expansion. If only very large frequencies are used, Φ_{-1}^{33} will dominate the series, whereas for only small frequencies, the highest index terms will dominate.

As an illustration of the importance of the frequency range of the experimental measurements, we present two examples. Both cases correspond to propagation along a linearly applied stress of the form, $\sigma_{33}(a_2) = c_0 + d_0 a_2$, with $c_0 = 62.5$ [MPa] and $d_0 = -62.5$ [MPa/mm]. We then numerically calculated the change in phase as a function of frequency over two ranges: 0.2–20 MHz and 4.0–20 MHz. This corresponds to what would have been measured experimentally. In addition, we calculated the $\delta\phi^{33}(\omega)$ curves, over the same two frequency ranges, for a number of other uniaxial linear distributions of the form $\sigma_{33}(a_2) = c + da_2$ for constants c and d in a suitable range, including the previously run values of $c_0 = 62.5$ [MPa] and $d_0 = -62.5$ [MPa/mm]. These $\delta\phi^{33}(\omega)$ curves were then compared to the original curve by summing the squares of the differences between the two curves at each frequency in the appropriate interval. When c and d equal 62.5 [MPa] and -62.5 [MPa/mm], respectively, the error thus calculated will be zero. For $c \neq 62.5$ [MPa] and $d \neq -62.5$ [MPa/mm], however, the $\delta\phi^{33}(\omega)$ curves will in general differ and thus give rise to a non-zero error. The resulting error surfaces, normalized to unity maximum, are plotted for the two frequency ranges in Figs 2 and 3 respectively.

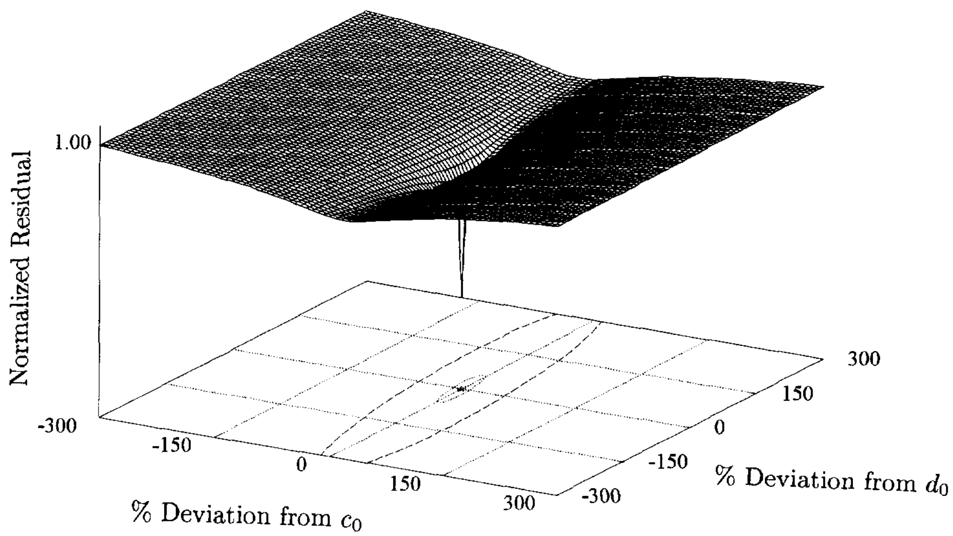

Fig. 2. Sum of square residual surface for propagation of a surface wave on a solid with initial stress which varies as $\sigma_{33}(a_2) = 62.5$ [MPa] -62.5 [MPa/mm]. $c_0 = 62.5$ is the correct value of surface stress, and $d_0 = -62.5$ [MPa/mm] is the correct value of surface stress gradient. Frequency range used in summing residuals was 0.2–20.0 MHz.

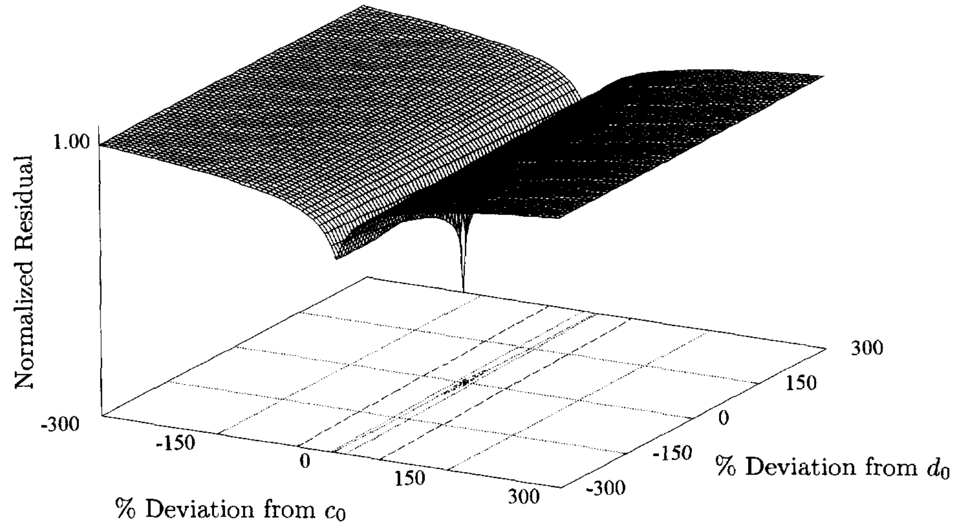

Fig. 3. Same as Fig. 2, but frequency range used in summing residuals was 4.0–20.0 MHz.

On examining Fig. 2 it can be seen that, if a large enough frequency range is included in the measurement, unique values of c and d can be obtained, since even slight changes in c and/or d from c_0 and d_0 will cause the resulting $\delta\phi^{33}(\omega)$ curves to change as well. Figure 3 illustrates, however, that if too small a frequency range is included, the error surface begins to develop a valley of relative minima, meaning that, over the measured frequency range, there are a loci of values (c, d) which give essentially the same $\delta\phi^{33}(\omega)$ curves. In this case, while the value of surface stress c is still determined fairly accurately, its gradient d can vary wildly depending on its initial guess. This state of affairs resulted from the fact that we removed the low end of the frequency information in generating Fig. 3 from Fig. 2, and is a manifestation of the fact that the remaining high frequency information is more sensitive to the surface stress than to its gradient. Of course, this is physically due to the fact that the Rayleigh wave fields become confined near the surface for large frequencies.

We close this section with the observation that although the phase change information $\delta\phi^{33}(\omega)$ was used to calculate the coefficients of the stress, σ_n^{33}, one could equally well have used $\delta\phi^{13}(\omega)$. That is, one could use the phase change information for a wave which propagates perpendicularly to the applied stress with a corresponding change in the constants which appear in the formulae $\alpha_i^{\parallel} \to \alpha_i^{\perp}$.

Biaxial stress states

For the uniaxial stress state examined above, it was found that measurement of the change of phase of the Rayleigh wave as a function of frequency for propagation along (or perpendicular to) the direction of applied stress was sufficient, in theory, to determine the stress state, provided the stress distribution admitted a representation in the form of eqn (14). In this section it will be seen that this is not true for biaxial stress states. Additional information will be needed in this case.

Assume that, in addition to $\sigma_{33}(a_2)$ there exists in the medium a stress $\sigma_{11}(a_2)$. The two distributions together define a state of bi-axial stress in the medium. Because the change in phase is a linear functional of the applied stress, the effect of this bi-axial stress state is the sum of the effects of each stress state individually. In the notation explained previously, we can write

$$\delta\phi^{3(1+3)}(\omega) = \delta\phi^{31}(\omega) + \delta\phi^{33}(\omega) \tag{23a}$$

$$\delta\phi^{1(1+3)}(\omega) = \delta\phi^{11}(\omega) + \delta\phi^{13}(\omega). \tag{23b}$$

From eqns (23), (6) and (10), the change of phase of a Rayleigh wave propagating in the a_3 direction would be given by

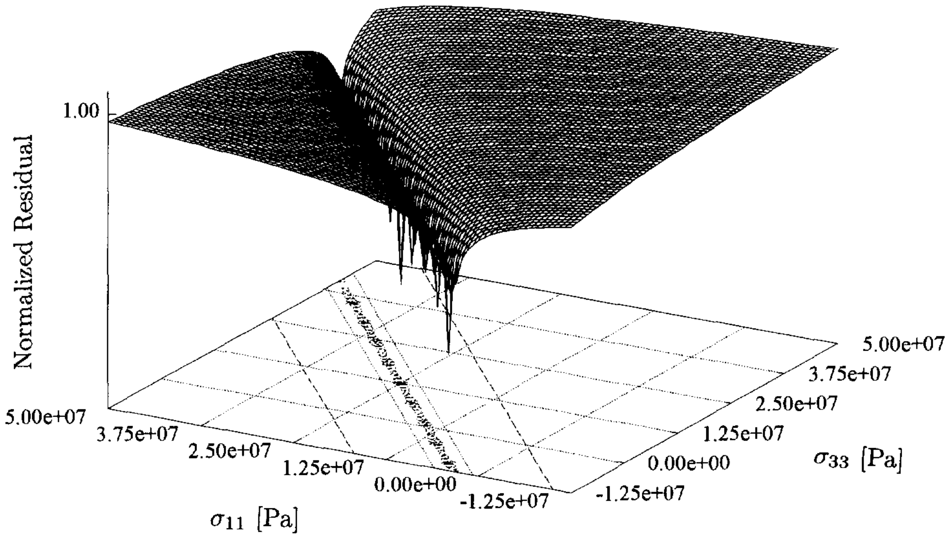

Fig. 4. Sum of square residuals surface for propagation of a surface wave in the a_3 direction on a solid with initial, uniform biaxial stress with principal stresses $\sigma_{11} = 25.0$ [MPa], $\sigma_{33} = 12.5$ [MPa].

$$-\frac{4(P/\omega)\delta\phi^{3(1+3)}(\omega)}{L_0} = \alpha_i^{\parallel} \sum_{n=0}^{N} \frac{C_{in}\sigma_n^{33}}{\omega^{n-1}} + \alpha_i^{\perp} \sum_{n=0}^{N} \frac{C_{in}\sigma_n^{11}}{\omega^{n-1}}$$

$$= \sum_{n=0}^{N} \frac{(\alpha_i^{\parallel}\sigma_n^{33} + \alpha_i^{\perp}\sigma_n^{11})C_{in}}{\omega^{n-1}}. \tag{24}$$

If the experimentally obtained $\delta\phi^{3(1+3)}(\omega)$ data are fit to an expansion of the form

$$-\frac{4(P/\omega)\delta\phi^{3(1+3)}(\omega)}{L_0} = \sum_{n=-1}^{N-1} \frac{\Phi_n^{3(1+3)}}{\omega^n} \tag{25}$$

and the terms of like powers in ω are compared in (24) and (25), there results:

$$\Phi_{n-1}^{3(1+3)} = (\alpha_i^{\parallel}\sigma_n^{33} + \alpha_i^{\perp}\sigma_n^{11})C_{in}. \tag{26}$$

The single set of coefficients, $\Phi_n^{3(1+3)}$, are seen from eqn (26) to be insufficient to obtain both sets of coefficients, σ_n^{33} and σ_n^{11}. Mathematically, the reason is simple; there are two unknowns and only one equation for any given n. Physically, this occurrence is due to the fact that the change of phase is affected by the stress both parallel and perpendicular to the propagation direction, albeit with different sensitivities.

Figure 4 is a numerical example illustrating the effect of varying the values of σ_{11} and σ_{33} on the change of phase curve, $\delta\phi^{3(1+3)}(\omega)$. Shown is the sum of the square residuals, as described earlier in conjunction with Fig. 2, for the case of a uniform, bi-axially applied stress field of $\sigma_{11} = 25$ [MPa] and $\sigma_{33} = 12.5$ [MPa]. The ordinate corresponds to the sum of the (square of the) differences between the correct change of phase curve obtained for $\sigma_{11} = 25.0$ [MPa] and $\sigma_{33} = 12.5$ [MPa] and the change of phase curve obtained using the σ_{11} and σ_{33} values corresponding to the ordinate. Due to the homogeneous nature of the initial deformation, the change of phase is actually frequency independent for any propagation direction.

As can be inferred from the prominent valley of minima in Fig. 4, the change of phase as a function of frequency is unaltered even if σ_{11} and σ_{33} deviate from their correct values of 25 and 12.5 MPa respectively, so long as the relation $\beta_1\sigma_{11}+\beta_2\sigma_{33} = \text{const}$ (for constants β_1 and β_2) is satisfied. The constants β_1 and β_2 which solely determine the slope of the valley in the $\sigma_{11}-\sigma_{33}$ plane are dependent only upon the second and third order elastic constants of the medium. It should be noted that the apparent spikes of minima along the general

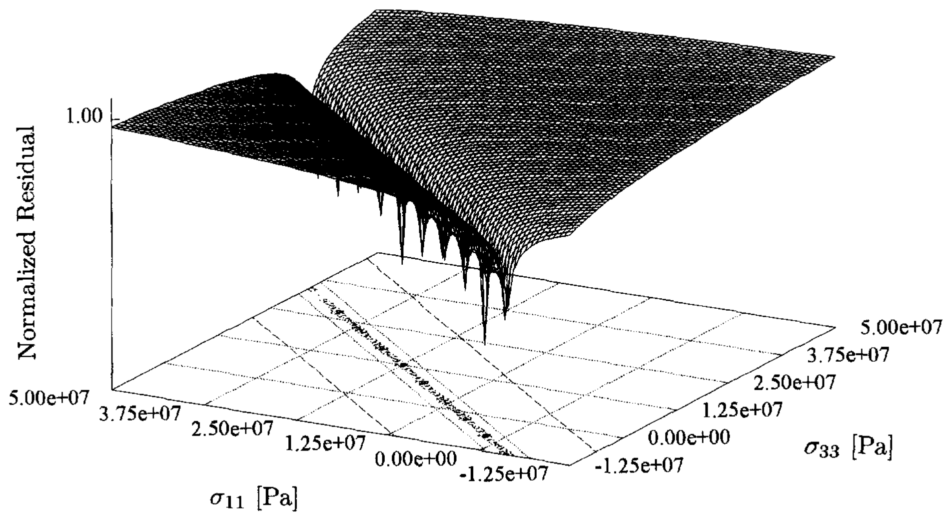

Fig. 5. Same as Fig. 4, but for propagation of the surface wave in the a_1 direction.

valley of minima are a result of the grid selection for plotting. The actual error will be zero along the entire line defined by $\beta_1\sigma_{11}+\beta_2\sigma_{33}$ = const.

To determine the change of phase which would be experienced by a Rayleigh wave propagating in the a_1 direction on the bi-axially stressed medium, one only has to interchange the roles of the constants $\alpha_i^{\parallel}$ and $\alpha_i^{\perp}$ in eqn (24). This results in

$$-\frac{4(P/\omega)\delta\phi^{1(1+3)}(\omega)}{L_0} = \sum_{n=0}^{N} \frac{(\alpha_i^{\perp}\sigma_n^{33}+\alpha_i^{\parallel}\sigma_n^{11})C_{in}}{\omega^{n-1}}, \tag{27}$$

and the coefficients in the expansion of $-4P\delta\phi^{1(1+3)}/\omega L_0$ similar to eqn (25) would yield the additional relations

$$\Phi_{n-1}^{1(1+3)} = (\alpha_i^{\perp}\sigma_n^{33}+\alpha_i^{\parallel}\sigma_n^{11})C_{in}. \tag{28}$$

One can see by comparing eqns (26) and (28) that the effect of propagating the Rayleigh wave in the a_1 direction is to switch the multipliers $\alpha_i^{\parallel}$ from σ_{33} to σ_{11} since σ_{11} is then parallel to the applied stress. There is a corresponding switch of the $\alpha_i^{\perp}$ multipliers from σ_{11} to σ_{33}.

The same conclusions which were drawn concerning the dependence of $\delta\phi^{3(1+3)}(\omega)$ on σ_{11} and σ_{33} (for propagation along a_3) hold for $\delta\phi^{1(1+3)}(\omega)$ and propagation along a_1. The corresponding sum of residual plots is shown in Fig. 5. In this case, the equation of the valley of minima in the σ_{11}-σ_{33} plane is given by $\beta_2\sigma_{11}+\beta_1\sigma_{33}$ = const, i.e., the coefficients are merely switched. This represents a 'reflection' (or mirror image) of the original valley of minima through the plane containing the 45 degree line of the σ_{11}-σ_{33} plane. This phenomenon can be seen by comparing Figs 4 and 5. The non-uniqueness of $\delta\phi^{1(1+3)}$ to individual values of σ_{11} and σ_{33} is also evident from this plot.

While the change of phase data from propagation in the a_3 or a_1 directions alone are not sufficient to reconstruct the coefficients of the biaxial stress state, a combination of the two sets of data, represented by eqns (26) and (28), gives a set of two equations in two unknowns for any value of n. The equations can be written in matrix form,

$$\begin{bmatrix} Q_n^{\parallel} & Q_n^{\perp} \\ Q_n^{\perp} & Q_n^{\parallel} \end{bmatrix} \begin{Bmatrix} \sigma_n^{33} \\ \sigma_n^{11} \end{Bmatrix} = \begin{Bmatrix} \Phi_{n-1}^{3(1+3)} \\ \Phi_{n-1}^{1(1+3)} \end{Bmatrix}, \tag{29}$$

where $Q_n^{\parallel} \equiv \alpha_i^{\parallel}C_{in}$ and $Q_n^{\perp} \equiv \alpha_i^{\perp}C_{in}$. These equations can be solved to yield the expansion coefficients of the stresses in terms of the fitted coefficients $\Phi_n^{3(1+3)}$ and $\Phi_n^{1(1+3)}$, provided

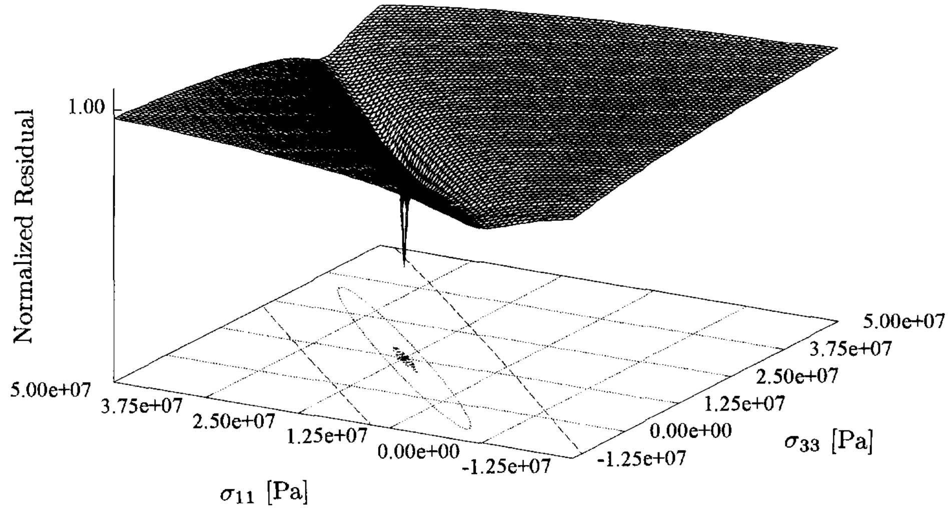

Fig. 6. Combined sum of square residual surface for surface wave propagation along a_1 and a_3 directions on a solid with uniform biaxial stress distribution with principal stresses $\sigma_{11} = 25.0$ [MPa], $\sigma_{33} = 12.5$ [MPa].

$(Q_n^{\perp})^2 \neq (Q_n^{\parallel})^2$. If this condition is met, the stress expansion coefficients follow from (29) as

$$\sigma_n^{33} = \frac{Q_n^{\parallel}\Phi_{n-1}^{3(1+3)} - Q_n^{\perp}\Phi_{n-1}^{1(1+3)}}{(Q_n^{\parallel})^2 - (Q_n^{\perp})^2}, \tag{30a}$$

$$\sigma_n^{11} = \frac{Q_n^{\parallel}\Phi_{n-1}^{1(1-3)} - Q_n^{\perp}\Phi_{n-1}^{3(1+3)}}{(Q_n^{\parallel})^2 - (Q_n^{\perp})^2}. \tag{30b}$$

In principle, eqns (30a) and (30b) give the required expansion coefficients for the stress distribution in the sample.

As an illustration of the effect of using information from propagation in both the a_1 and a_3 directions, we present Fig. 6. This sum of residual surface, which results from simultaneously computing the errors in $\delta\phi^{3(1+3)}$ and $\delta\phi^{1(1+3)}$ due to changing values of σ_{11} and σ_{33} from 25.0 and 12.5 [MPa] respectively, is simply the sum of the previous two surfaces, re-normalized to unity maximum. As can be seen from this surface, there is now a single, unique minimum for which the deviation in the change of phase curves will vanish. This shows numerically that the inversion will be sensitive to changes in either or both values of stress and, therefore, that the inversion will, in principle, be unique. Of particular noteworthiness is the lack of any adjacent local minima over the relatively large range of σ_{11} and σ_{33} displayed in the figure. Although the numerical example presented was for a uniform biaxial stress, it follows from eqns (30a,b), that the inversion will be unique for stress fields which vary as any power of a_2 or linear combinations of such functions so long as the condition, $(Q_n^{\perp})^2 \neq (Q_n^{\parallel})^2$ is satisfied.

DISCUSSION AND CONCLUSIONS

As a closing note, we give a physical interpretation to the condition $(Q_n^{\perp})^2 = (Q_n^{\parallel})^2$ which would cause the above results to break down for that particular n. By examining the definitions of $Q_n^{\parallel}$ and $Q_n^{\perp}$ it can be seen that the condition will occur if the effect of a stress applied perpendicular to the propagation direction (i.e., $Q^{\perp}$) has the same effect on the change of phase of the wave as a stress applied in the direction of propagation (i.e., $Q^{\parallel}$). A sufficient, although not necessary condition for this to happen is for $\alpha_i^{\perp} = \alpha_i^{\parallel}$ for all $i \in \{1, \ldots, 5\}$. This condition was checked for several alloys of aluminum and for mild steel. It was not satisfied for any of the materials checked. We also note that all of the results presented here should be considered strictly formal at this point since there is little or no experimental data upon which to test the robustness of the inversion algorithms to experimental errors.

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APPENDIX

Constants f_{ij} appearing in eqn (7).

$$f_{11} = (\kappa_s/V_o)^2\,; \quad f_{12} = (\kappa_l K_2/V_o)^2\,; \quad f_{13} = -2\kappa_s\kappa_l K_2/V_o^2,$$

$$f_{21} = f_{11}\,; \quad f_{22} = K_4^2 f_{21}\,; \quad f_{23} = -2K_4 f_{21},$$

$$f_{31} = -2f_{11}\,; \quad f_{32} = \kappa_l K_2 K_4 f_{31}/\kappa_s\,; \quad f_{33} = -[K_4+\kappa_l K_2/\kappa_s]f_{31},$$

$$f_{41} = 1/V_o^4+\kappa_s^4\,; \quad f_{42} = K_2^2/V_o^4+(\kappa_s\kappa_l K_4)^2\,; \quad f_{43} = -2[K_2/V_o^4+\kappa_s^3\kappa_l K_4],$$

$$f_{51} = 2f_{11}\,; \quad f_{52} = \kappa_l K_2 K_4 f_{51}/\kappa_s\,; \quad f_{53} = -[\kappa_l K_4/\kappa_s+K_2]f_{51}.$$